# Phase Transition Phenomena in Wireless Ad Hoc Networks

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***Abstract*— There are many contexts in distributed wireless networks where there is a critical threshold, corresponding to a minimum amount of the communication effort or power expenditure by individual nodes, above which a desirable global property exists with high probability. When this individual node effort is below the threshold the desired global property exists with a low probability. This "phase transition" is typically seen to become sharper as the number of nodes in the network increases. We discuss in this paper some examples of properties that exhibit such critical behavior: node reachability with probabilistic flooding, ad-hoc network connectivity, and sensor network coordination. We discuss the connections between these phenomena and the phase transitions that have been shown to arise in random graphs. We argue that a good understanding of these phase transition phenomena can provide useful design principles for engineering distributed wireless networks.**

## I. Introduction

Research on distributed multi-hop wireless networks, otherwise referred to as wireless ad hoc networks [1], has evolved from the early 1970's DARPA packet radio program [2]. There has been a renewed interest in this field as inexpensive, energy-efficient, and miniaturized wireless technologies are beginning to mature and take hold commercially. Wireless ad hoc networks, which can be deployed rapidly as they do not require much existing infrastructure, are expected to find applications in a number of diverse settings. Examples include sensor networks, disaster recovery, law enforcement, military communications, distributed computing, wireless LANs, and special events such as conferences and festivals.

Recent work [3], [4] has shown that if we assume each node in an ad-hoc network has constant power, there is a critical transmission power required to ensure with high probability that two nodes in the network can communicate with each other through multi-hop paths. It is desirable to minimize the energy consumption of the wireless nodes, but if the node transmission powers are decreased below the critical level, there is a precipitous drop in the connectivity of the network. We show in this paper that similar "phase transitions" occur in other situations as well in distributed wireless networks. From an engineering standpoint, it is crucial that we study and understand these phase transitions because we would like to design systems which operate just beyond the critical point, where there is efficient resource utilization.

Such behavior has been known to mathematicians for several decades in the form of "zero-one" laws in Bernoulli Random Graphs. The basic idea is that for certain monotone graph properties such as connectivity, as we vary the average density of the graph, the probability that a randomly generated graph exhibits this property asymptotically transitions sharply from zero to one at some critical threshold value.

The rest of the paper is organized as follows: in section II, we discuss two models for generating random graphs - one that has been extensively studied earlier, and one that is more relevant to an ad-hoc wireless network. We present some known results on phase transitions from random graph theory and discuss their applicability to the fixed transmission radius model in section III. In sections IV, V, and VI, respectively, we present three contexts where phase transitions arise: ad-hoc network connectivity, coordination in a sensor network, and probabilistic flooding for route discovery. We conclude with a discussion of the results and future work in section VII.

## II. Random Graph Models

A random graph G can be loosely described as a graph that is generated by some prescribed random experiment. The description of the random experiment constitutes a *random graph model*. In most cases these models contain a tuning parameter which varies the average density of the random graph constructed. Here are some examples of random graph models:

- Fixed edge number model: $G = G(n, e)$, given $e$ and $n$, choose $G$ uniformly at random from all graphs consisting of $n$ vertices and $e$ edges.

- Bernoulli model: $G = G(n, p)$, given $n$, and $p$, construct $G$ with $n$ vertices such that there is an edge between any two pairs of nodes with probability $p$.

- Fixed radius model: $G = G(n, R)$, given $n$ points placed randomly according to some distribution in the Euclidean plane, construct $G$ with $n$ vertices corresponding to these points in such a way that there is an edge between two vertices $v$ and $w$ if and only if the corresponding points are within a distance $R$ of each other.

- Dynamic model: $G = G(n, t)$, given $n$ points, at each discrete time step add a randomly chosen edge to the graph, i.e. add the $t^{th}$ edge at time $t$.

We describe two of these models in further detail in the subsections below. The first, the Bernoulli Random Graph (BRG) model is the one that has been considered by mathematicians and there is considerable literature on this subject [6]. The second, the Fixed Radius (FR) model is one that most naturally models the communication graphs formed in wireless networks where each node has a fixed transmission radius.

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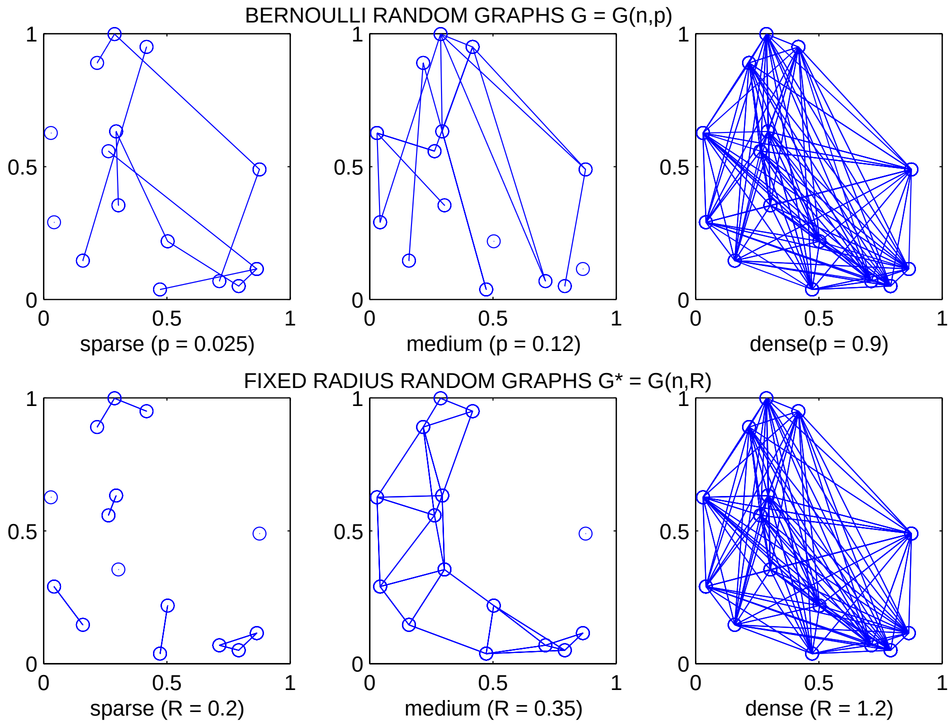


Fig. 1. Typical Random Graphs with Varying Degrees of Connectivity Generated using the Bernoulli and Fixed Radius Models (n=15)

### A. *Bernoulli Random Graphs*

As described above, one can imagine a single Bernoulli random graph $G = G(n,p)$ being generated by an experiment where we take the $n$ vertices and flip a coin for each pair of vertices to decide if there should be an edge between them. When $p = 0$, the resulting random graph has no edges, while when $p = 1$, we always get the complete graph $K_n$. As the parameter $p$ increases from 0 to 1, the average density of the random graphs increases (see figure 1, top row).

### B. *Fixed Radius Model*

Consider an ad-hoc wireless network with $n$ nodes, located randomly in some service area, each of which is assumed to transmit with a fixed radio power in an idealized environment where it can be heard by other nodes that are within some radius $r$. Thus two nodes can communicate directly with each other if an only if they are no more than a distance $r$ apart. We can now talk about the underlying communication graph $G^* = G^*(n,R)$ in this fixed radius (FR) model - at a given instant let each node in the network correspond to a unique vertex in $G^*$, with an edge between a pair of vertices if the corresponding nodes can communicate with each other. The lower row of figure 1 illustrates sample random graphs generated according to this model. It is assumed that the service area is a square with unit sides. As the parameter $R$ increases from $0$ to $\sqrt{2}$, the average density of the random graphs increases.

A random graph generated using the fixed radius model has some different characteristics than the Bernoulli random graph. Consider three nodes $i, j$, and $k$. In the fixed radius model, "the event that there are links between $i$ and $j$ and $j$ and $k$ is not independent of the event that there is a link between $i$ and $k$ ... as the former is true given the latter only if $j$ lies in the intersection of two discs of radius $[R]$ centered at $i$ and $k$" [3]. Still, they do have some things in common. In both models, there is a parameter which varies the density of the graphs produced. We will show in later sections examples where phase transitions with respect to the parameter $p$ observed in the Bernoulli graph model also show up with respect to the parameter $R$ in the fixed radius model. It may thus be possible to apply some results from the literature on Bernoulli random graphs to fixed radius random graphs that are of interest to us in the context of wireless ad-hoc networks.

## III. Random Graphs and Phase Transitions

To quote Bollobás : "one of the main aims of the theory of random graphs is to determine when a given [graph] property is likely to appear... Erdös and Rényi were the first to show that most monotone properties appear rather suddenly. In rather vague terms, a threshold function is a critical time, before which the property is unlikely and after which it is likely" [6].

Let us consider one example of zero-one laws for random graphs which applies for first order graph properties. First order properties of graphs are those that can be described using

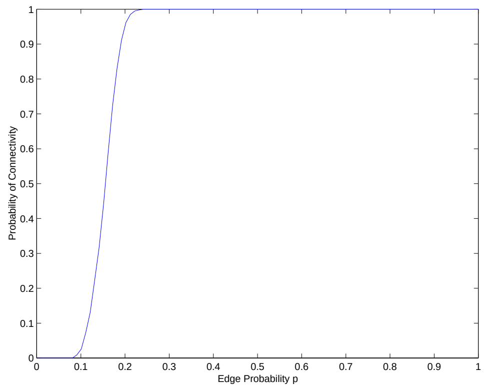


Fig. 2. Phase Transition in Probability of Connectivity in the Bernoulli Random Graph Model (n=15)

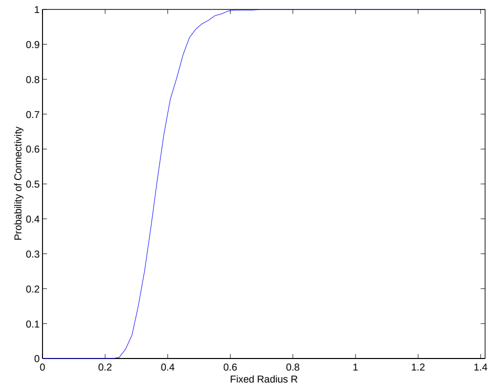


Fig. 3. Phase Transition in Probability of Connectivity in the Fixed Radius Ad-hoc Wireless Network (n=15)

a language consisting of the basic Boolean logic connectives ($\wedge, \vee, \neg$), existential and universal quantifiers ($\exists, \forall$), variables depicting nodes, equality and adjacency (written $I(x, y)$). Examples of first order properties are a) "there are no isolated points" ($\forall\ x,\ \exists y\ s.t.\ \bar{I}(x, y)$), and b) "contains a triangle" ($\exists(x, y, z)\ s.t.\ I(x, y)\ \wedge I(x, z)\ \wedge I(y, z)$).

*Theorem 1:* For every first order graph property A, $\lim_{n\to\infty} Pr[G(n, p)\ has\ A] = 0\ or\ 1$.

A proof of this theorem can be found in [5].

If the property $A$ is monotone with respect to the addition of edges, then $Pr[G(n, p)\ \ has\ \ A]$ is also monotone. Thus asymptotically, for large $n$, the probability of a random graph property undergoes a sharp zero-one transition for some critical edge parameter value $p = p_{crit}$.

We conjecture that *properties which satisfy a zero-one law for Bernoulli Random Graphs also satisfy a zero-one law for the Fixed Radius model*.

## IV. Network Connectivity

Although the property "the graph is connected" is not a first order property, it also undergoes a similar zero-one transition in a Bernoulli random graph [6]. This is shown in figure 2. Similar experimental work showing phase transition behavior in BRG is described in [7]. Among other things, the results in that paper demonstrate that the transitions become sharper as $n$ becomes larger.

In [3], the authors proved that such transitions also take place in an ad-hoc network with fixed transmission powers for all nodes where the communication graph can be described as a fixed radius random graph. Sanchez et al. showed in [4] that this behavior is in fact robust with respect to different mobility models. In other words, even when the nodes are moving around, there is still a critical transmission range that is required to ensure connectivity in the network.

## V. Coordination in a Sensor Tracking Network

Consider the following scenario: there are some wireless Doppler radar sensors with some computational resources distributed in some location, each of which can communicate directly with only a subset of other sensors. There are some moving targets in the area whose position and velocity these sensors are required to track. In order to obtain this tracking information it is required that three sensors co-ordinate directly with each other in a distributed manner to track each target. What should the minimum communication level be in this system to ensure that all targets can be tracked?

Specifically, let us assume there are $k$ distinct targets and $n = 3k$ radar sensors. Each target is visible to each sensor. We want to know if we can assign three mutually communicating sensors for tracking each target. Observe that this can be done if and only if the underlying communication graph can be partitioned into triangles. The following theorem tells us something about the probability that a random graph can be partitioned into triangles.

*Theorem 2:* A = "There exists a partition into triangles" is a monotone, first-order graph property.

Proof: A is clearly monotone - if there exists a partition into triangles with some graph G, additional edges do not remove this property. Further, it can be expressed in the first order language of graphs as follows:

$$\forall x, \{\ \exists(y, z)\ s.t.\ \{I(x, y) \wedge I(y, z) \wedge I(x, z)\}\ \}\ \wedge \{\ \forall x'\ s.t.\ \{I(x, x') \wedge \neg(x' = y, z)\}, \exists(y', z')\ s.t.\ \{\neg\ (y', z' = x, y, z) \wedge I(x', y') \wedge I(y', z')\ \wedge I(x', z')\}\ \} \square$$

*Corollary 1:* The probability that there exists a partition of a Bernoulli random graph $G(n, p)$ into triangles will show an asymptotically sharp phase transition from 0 to 1 with respect to the edge probability $p$.

Proof: This is an immediate consequence of Theorems 1 and 2. $\square$

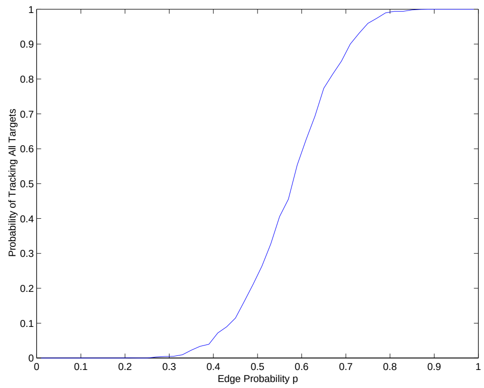


Fig. 4. Phase Transition in Probability of Tracking All Targets for Sensors with a Bernoulli Random Graph Model for Communication (n=9)

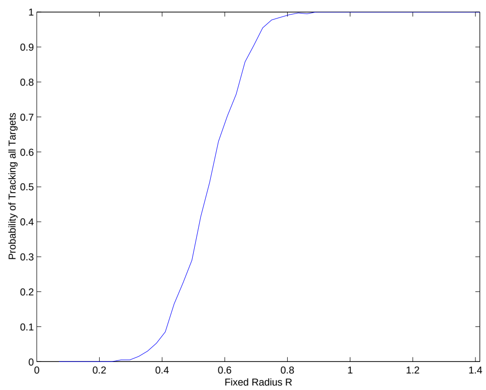


Fig. 5. Phase Transition in Probability of Tracking All Targets for Sensors with a Fixed Radius Model for Communication (n=9)

Figures 4 and 5 present experimental results that show this phase transition when the underlying communication graph is modeled by the two random graph models. Thus again it appears there is a critical value of communication level/ transmission power which is required to ensure that the desired coordinated tracking can take place. We illustrate in the next section another phase transition that arises in the context of route discovery in an ad-hoc network.

## VI. Reachability in Probabilistic Flooding Schemes

*Gossiping*, a technique originally proposed for probabilistic multicast in communication networks [9], has recently been applied to minimize query traffic while flooding in reactive on-demand routing protocols [8]. The basic idea is as follows: the source node initiates a flood of route query packets for the purposes of discovering a route to the destination. It is desired that the route query packets be forwarded to all nodes in the network so that the destination can respond to it. In basic flooding, each node always forwards the query packet to all its neighbors when it receives it. In probabilistic flooding, the source initiates the query and all other nodes that receive a query packet will transmit in order to forward it to their neighbors with some

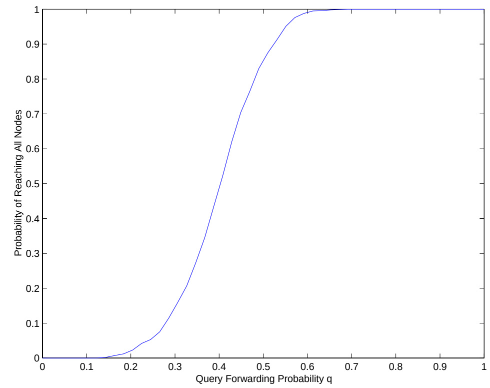


Fig. 6. Phase Transition in Probability of Reaching All Nodes in a Probabilistic Flooding Scheme (n=15, FR model with radius R = 0.5)

probability $q$ and discard the packet with probability $1 - q$.

Experimental results suggest that there is a critical value of the query forwarding probability $q$ that is required to ensure with high probability that all nodes receive the route query. The corresponding phase transition is shown in figure 6. We note that the critical value of $q$, the value beyond which flooding succeeds with very high probability, in itself would depend upon the transmission range / average node-degree of the underlying communication graph. As the number of neighbors that each node has increases the critical value of $q$ decreases, as is to be expected. Thus there is an interesting trade-off in this situation: if the transmission radius is large, more power is expended, but the query traffic is minimized, whereas if the transmission radius is small then less power is expended by each node, but the number of route query packets will increase as the critical value of $q$ increases.

## VII. Discussion and Conclusions

We have presented in this paper some examples of phase transitions that arise in wireless ad-hoc networks, and discussed briefly the related theory of Bernoulli random graphs which can aid us in proving when such phase transitions will occur.

What is the significance of phase transition behavior in wireless ad-hoc networks from a system design perspective? The key is to notice that in all these examples, the $x - axis$ represents the utilization of some resource. In the case of network connectivity and the radar sensor network we were concerned with the transmission power of each node. This is, of course, directly linked to the consumption of energy at each node. In the case of the probabilistic flooding scheme we were concerned with the probability $q$ of query-forwarding. When $q$ is high there is a correspondingly high number of query packets that will be generated in the network which incur a bandwidth overhead cost.

We would like to engineer wireless networks to be just to the right of the phase transition in each of these cases. This represents the point where the resource utilization for the individual nodes is minimized while the desired global property is realized with high probability. A greater understanding of such transitions provides us with valuable engineering design principles.

## VIII. Acknowledgment

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