Recap

Phy Layer
- how to model wireless channels
- path loss, random varying, coherence time/dynamics
- delay spread/ISI
- Comm schemes:
  - modulation
  - MIMO, CDMA/DSSS, FHSS, Coding, OFDM

Power
- Error Rate
- Throughput
- Tradeoff

Link Layer
- Power Allocation over parallel channel (waterfilling)
- Power control over independent links.
Power Allocation to max sum rate over parallel chns subject to total power constraint

\[
\begin{align*}
    \max & \quad \sum_{c=1}^{N} \log \left( 1 + \frac{P_i \cdot g_i}{n_i} \right) \\
    \text{s.t.} & \quad \sum_{i=1}^{N} P_i \leq P_T \\
    & \quad P_i \geq 0 \quad \forall i
\end{align*}
\]

CONstrained OPT. PROBLEM

Rate (modulation scheme) vs SNR

SNR = \frac{P_i \cdot g_i}{n_i}
How to solve optimization problems involving multiple variables and inequality constraints.

Problem: \( \max_{\vec{x}} f(\vec{x}) \)

\( \vec{x} = [x_1, \ldots, x_N] \)

s.t. \( g_i(\vec{x}) \leq c_i \) \( \quad i = 1, 2, \ldots, M \)

Solution Approach: KKT conditions

\[ L(\vec{x}, \vec{\lambda}) = f(\vec{x}) - \sum_{i=1}^{M} \lambda_i (g_i(\vec{x}) - c_i) \]

Lagrangian

Find: \( \vec{x}^*, \vec{\lambda}^* \) s.t.

1. \( \frac{\partial L}{\partial x_i} = 0 \quad i \neq \star 
2. \lambda_i^* \geq 0 \quad i \neq \star 
3. g_i(\vec{x}^*) \leq c_i \quad i \neq \star 
4. \lambda_i^* [g_i(\vec{x}^*) - c_i] = 0 \quad i \neq \star 

\underline{Complementary slackness condition}
\[ \nabla_i \sum g_i(x) - c_i = 0 \]

\[ \Rightarrow \begin{cases} \nabla_i^* = 0 \\ g_i(x^*) - c_i = 0 \end{cases} \]

Example:

\[ f(x) = -x^2 \]

\[ g(x) = x \quad c = a \]

\[ \max f(x) \text{ s.t. } x \leq a \]

\[ \max -x^2 \text{ s.t. } x \leq a \]

\[ L(x, \lambda) = -x^2 - \lambda(x - a) \]

1. \[ \frac{\partial L}{\partial x} = 0 \Rightarrow -2x - \lambda = 0 \Rightarrow \lambda = -2x \]
2. \[ x^* \geq 0 \]
3. \[ x^* \leq a \]
4. \[ \text{either } x^* = 0 \text{ or } x^* = a \]
Let's consider each case:

1. $a < 0 \Rightarrow x^a = a$
2. $a = 0 \Rightarrow x^a = 0$
3. $a > 0$ either $x^a \leq 0$ or $x^a > 0$

Back to our problem! A constraint

$$P_i \rightarrow x_i$$
$$f(x) = f(\Phi) = \sum_{i=1}^{N} \log \left(1 + P_i \frac{x_i}{\lambda_i}\right)$$

Variables: $N$

$$\lambda_i$$

$$g_1(x) = f(\Phi) = \sum_{i=1}^{N} P_i \leq P_f$$
$$g_{2(\Phi)} \ldots g_{N+1}(x): P_i \geq 0 \quad \forall i: 1 + N$$

$$L(\bar{P}, \lambda) = \sum_{i=1}^{N} \log \left(1 + P_i \frac{x_i}{\lambda_i}\right) - \lambda_1 (\Sigma P_i - P_f)$$
$$- \sum_{i=1}^{N} \lambda_{i+1} (-P_i)$$
$$+ \sum_{i=1}^{N} \lambda_{i+1} (-P_i) = \Sigma \lambda_i \cdot P_i$$

$\lambda_1 = M \quad \lambda_{i+1} = \lambda_i$
\[ \vec{\lambda} = \left[ \lambda_1, \lambda_2, \ldots, \lambda_{N+1} \right] \]
\[ \left( \vec{\lambda}, \vec{\mu}, \vec{v} \right) = \left[ \mu_1, \mu_2, \ldots, \mu_N \right] \]

\[ L(\vec{\rho}, \vec{z}) = L(\vec{\rho}, \mu, \vec{v}) = \frac{1}{N} \sum_{i=1}^{N} \log \left( 1 + \rho_i \frac{z_i}{\mu_i} \right) - \mu \left( \sum_{i=1}^{N} \rho_i - \rho \right) + \sum_{i=1}^{N} v_i \rho_i \]

**KKT Conditions:**

1. \[ \frac{\partial L}{\partial \rho_i} = 0 \Rightarrow \frac{\rho_i}{\mu_i} \frac{z_i}{\mu_i} - \mu_i^* + v_i^* = 0 \]

2. \[ \mu_i^* \geq 0, \quad v_i^* \geq 0 \]

3. \[ \sum_{i=1}^{N} \rho_i \mu_i^* \leq \rho \quad \rho_i^* \geq 0 \]

4. Either \[ \mu_i^* = 0 \]
   or \[ \sum_{i=1}^{N} \rho_i = \rho \]

   focus on \[ i \text{ s.t. } \rho_i^* \neq 0 \]
   i.e. \[ v_i^* = 0 \]
   in this case \[ 1 \text{ simplifies to: } \]
   \[ \frac{\beta_i / \mu_i}{1 + \rho_i \beta_i / \mu_i} = \mu_i^* \]
\[ \mu^* = \frac{g_i / \mu_i}{1 + \mu^* \frac{g_i}{\mu_i}} \]

\[ 1 + \mu^* \frac{g_i}{\mu_i} = \frac{g_i / \mu_i}{\mu^*} \]

\[ \pi^* = \left( \frac{g_i / \mu_i}{\mu^*} - 1 \right) \cdot \frac{1}{(g_i / \mu_i)} \]

\[ = \frac{1}{\mu^*} - \frac{1}{(g_i / \mu_i)} = \]

\[ \pi^* = \left( \frac{1}{\mu^*} - \frac{z_i}{g_i} \right) \]

or

\[ \pi^* = 0 \]

meaner \[ \sum \pi^* = \pi_t \]

\[ \sum_{i=1}^{n} \left( \frac{1}{\mu^*} - \frac{z_i}{g_i} \right) = \pi_t \]

Together a, b, and the water filling solution.
In HW3:

\[ f = \log \left( 1 + \frac{P_i}{a_i} \right) \]

\[ \min \sum_{i=1}^{N} (R_i - T_i)^2 \]

subject to \( \sum P_i \leq P_T \)

smallest when \( R_i = T_i \)

\[ \min f \left( \right) = \max - f \left( \right) \]
Slotted Aloha - Randomized Access.

Early 1970s

Single channel shared by all transmitters.

When you have a packet, send it.

Prob. $\frac{1}{p}$, IID over time.

Equivalently: they always have a packet to send, but send with a randomized (IID) procedure.
n nodes, backlogged/Saturated

each node indep. transmits w/p

p = p. at each slot (iid over time)

Saturation

Throughput = ?

Collision if more than 1 pkt in each slot

\[ E[\text{Throughput}] = n p (1-p)^{n-1} \]

from n 1 time slot, since iid:

\[ \text{Pkt's:} \begin{cases} 0 \text{ pkts} & (1-p)^n \\ 1 \text{ pkt} & np(1-p)^{n-1} \\ > 1 \text{ pkts} & 1 - np(1-p)^{n-1} \end{cases} \]
For $n = 2$, \[ \max_2 p \frac{b_1 (1-b)^{n-1}}{p} \]

Maximized when $p^* = \frac{1}{2}$

\[ 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \leq \max \text{ throughput for 2 users.} \]

In general:

\[ \max_p n p (1-p)^{n-1} = T. \]

For fixed $n$:

\[ \frac{\partial T}{\partial p} = 0 \Rightarrow \frac{2 \cdot (n-1) \cdot (1-p)^{n-2}}{p} \bigg|_{p^*} = 0 \]

\[ (1-p)^{n-1} + p^{n} \cdot (n-1) \cdot (1-p)^{n-2} = 1 = 0 \]

\[ \Rightarrow (1-p)^{n-1} = p^n \cdot (n-1) \cdot (1-p)^{n-2} \]
\[ 1 - p^* = p^*(n-1) \]

\[ = 1 - p = p \cdot n - p \]

\[ p^* = \frac{1}{n} \]

max throughput: \( n \cdot \frac{\hat{p}(1 - \hat{p})^{n-1}}{\sqrt{n}} \)

\[ \frac{\hat{p}}{\sqrt{n}} \left( \frac{1}{1 - \frac{1}{n}} \right)^{n-1} \]

\[ n = 2 \quad \frac{1}{2} \quad T(n) \rightarrow \frac{1}{e} \]

\[ n = 3 \quad \frac{4}{9} \]  

\[ \sim 36.2 \% \]

max throughput of slotted Aloha

as \( n \to \infty \) (Clark)
What if different users send different rates?

$P_i$ - slotted Aloha:

2 users: $P_1, P_2$

$T_1 = P_1 (1 - P_2)$

$T_2 = P_2 (1 - P_1)$

maximum feasible saturation throughput regime
Key Result:

At the boundary of the 2 user pi-slotted Aloha saturation throughput region: \( P_1 + P_2 = 1 \).

-not an obvious fact!

Is it true that \( \sum_{i=1}^{n} P_i = 1 \) for any number of users to have a throughput vector at the boundary of the saturation throughput region?

[YES!]

Ti (T1, T2, T3) on the boundary corresponds to (P1, P2, P3)
\[ \text{s.t. } P_1 + P_2 + P_3 = 1 \]
More general result for 
p-i-slotted Aloha:

At the boundary of n-user saturation throughput region:

\[ E_k = 1 \]

\[ T_i = \frac{p_i}{1 - \prod_{j \neq i} (1 - p_j)} \]

\[ \vec{p} \rightarrow \frac{1}{T} \]

In terms, can now formulate:

\[ \max \ f(T_1, T_2, T_3) \]

s.t. \[ \sum E_k = 1 \]