EE597 Lecture 6 June 8, 2016

Recap

Phy Layer

- how to model Wireless channels

  path loss, random variations, coherence time / dynamics

  Delay spread / ISI

- Comm schemes:
  - modulation
  - MIMO, CDMA / DSSS, FHSS, Coding, OFDM

Power - Error Rate - Throughput

Tradeoff.

Link Layer

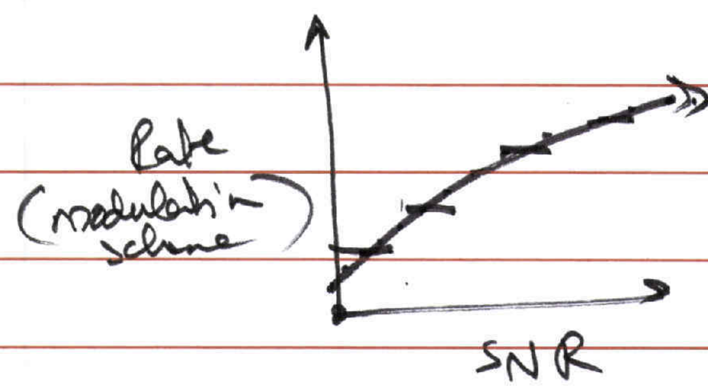
→ Power Allocation over parallel channel (Waterfilling)

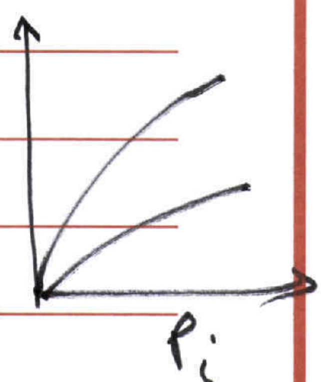
Power control over independent links.

Power Allocation to max sum rate over parallel chns subject to total power constraint

$$\boxed{\begin{array}{ll} \max & \sum_{i=1}^{N} \log\left(1 + P_i \cdot \frac{g_i}{n_i}\right) \\ \text{s.t.} & \sum_{i=1}^{N} P_i \le P_T \\ & P_i \geq 0 \quad \forall i \end{array}}$$

CONSTRAINED OPT. PROBLEM





$$SNR = P_i \cdot \frac{g_i}{N_i}$$

HOW TO SOLVE OPTIMIZATION PROBLEMS INVOLVING MULTIPLE VARIABLES AND INEQUALITY CONSTRAINTS.

$\vec{x} = [x_1 \dots x_N]$

Problem: $\max\limits_{\vec{x}} \; f(\vec{x})$

$$\text{s.t.} \quad \left.\begin{aligned} g_1(\vec{x}) &\le c_1 \\ g_2(\vec{x}) &\le c_2 \\ &\vdots \\ g_M(\vec{x}) &\le c_M \end{aligned}\right\} \text{ M inequality constraints}$$

Solution Approach: KKT conditions

$$\mathcal{L}_e(\vec{x}, \vec{\lambda}) = f(\vec{x}) - \sum_{i=1}^{M} \lambda_i \left(g_i(\vec{x}) - c_i\right)$$

Lagrangian

$$\Updownarrow$$

$$f(\vec{x}) + \sum_{i=1}^{M} \lambda_i \left(c_i - g_i(\vec{x})\right)$$

find: $\vec{\lambda}^*, \vec{x}^*$ s.t.

① $\left.\dfrac{\partial \mathcal{L}_e}{\partial x_i}\right|_{\vec{x}^*, \vec{\lambda}^*} = 0 \;\forall i$ ② $\lambda_i \ge 0 \;\forall i$ ③ $g_i(\vec{x}^*) \le c_i \;\forall i$

④ $\underbrace{\lambda_i^* \left[g_i(\vec{x}^*) - c_i\right] = 0 \;\forall i}_{\text{complementary slackness condition}}$

$$\lambda_i^* [g_i(\vec{x}) - c_i] = 0$$

$\Rightarrow$ either $\begin{cases} \lambda_i^* = 0 \\ \text{OR} \\ g_i(\vec{x}) - c_i = 0 \end{cases}$

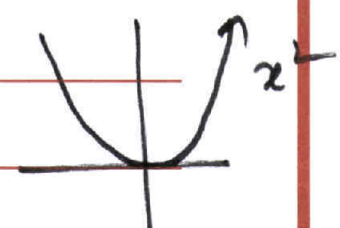
$$g_i(\vec{x}) = c_i$$ constraint is <u>tight</u>

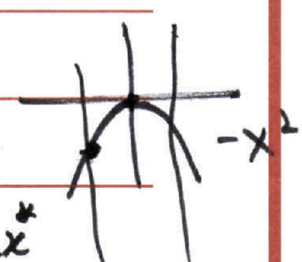
<u>example:</u> $f(x) = -x^2$

$g(x) = x \qquad c = a$

$\max f(x)$ s.t. $x \le a$

$\max -x^2$ <u>s.t. $x \le a$</u>





$$L(x, \lambda) = -x^2 - \lambda(x - a)$$

① $\left.\frac{\partial L}{\partial x}\right|_{x^*, \lambda^*} = 0 \Rightarrow -2x^* - \lambda^* = 0 \Rightarrow \lambda^* = -2x^*$

② $\lambda^* \ge 0$ ③ $x^* \le a$ ④ either $\lambda^* = 0$ <u>OR</u> $x^* = a$

Let's consider each case:

(1.) $a < 0 \Rightarrow x^* = a$

(2) $a = 0 \Rightarrow x^* = 0$

(3) $a > 0$ either $\boxed{x^* = 0}$ OR $\lambda^* < 0$ X

Back to our problem! # of variables: N

constraints $m = N+1$

$P_i \rightarrow x_i$

$$f(\vec{x}) = f(\vec{P}) = \sum_{i=1}^{N} \log\left(1 + P_i \frac{g_i}{n_i}\right)$$

$$\lambda_1 : g_1(\vec{x}) = g(\vec{P}) = \sum_{i=1}^{N} P_i \leq P_T$$

$$\lambda_2 \cdots \lambda_{N+1} \; g_2(\vec{x}) \ldots g_{N+1}(\vec{x}) : \quad P_i \geq 0 \quad \forall i: 1 \text{ to } N$$

$$\boxed{-P_i} \leq 0$$

$$\mathcal{L}(\underset{1\times N}{\vec{P}}, \underset{1\times N+1}{\vec{\lambda}}) = \sum_{i=1}^{N} \log\left(1 + P_i \frac{g_i}{n_i}\right) - \underbrace{\lambda_1 \left(\sum P_i - P_T\right)}_{\rightarrow \mu(\sum P_i - P_T)}$$

$$\underbrace{- \sum_{i=1}^{N} \lambda_{i+1} \cdot (-P_i)}$$

$$+ \sum_{i=1}^{N} \lambda_{i+1} (P_i) = \sum \nu_i \cdot P_i$$

$\lambda_1 = \mu \qquad \lambda_{i+1} = \nu_i$

$$\vec{\lambda} = [\lambda_1, \lambda_2 \cdots \lambda_{N+1}]$$
$$= [\mu, \nu_1 \cdots \nu_N]$$

$$\mathcal{L}(\vec{P}, \vec{\lambda}) = \mathcal{L}(\vec{P}, \mu, \vec{\nu}) =$$
$$\sum_{i=1}^{N} \log\left(1 + P_i \frac{g_i}{n_i}\right) - \mu\left(\sum_{i=1}^{N} P_i - P_T\right) + \sum_{i=1}^{N} \nu_i P_i$$

KKT conditions:

① $\left.\frac{\partial \mathcal{L}}{\partial P_i}\right|_{\vec{P}^*, \mu^*, \vec{\nu}^*} = 0 \Rightarrow \frac{\frac{g_i}{n_i}}{1 + P_i^* \cdot \frac{g_i}{n_i}} - \mu^* + \nu_i^* = 0$ — ①

N different equations

② $\mu^* \geq 0$, $\nu_i^* \geq 0$

③ $\sum_{i=1}^{N} P_i^* \leq P_T$ $\qquad P_i^* \geq 0$

④ either $\mu^* = 0$ OR $\sum_{i=1}^{N} P_i = P_T$ $\qquad$ either $\nu_i^* = 0$ or $P_i^* = 0$

focus on i s.t. $P_i^* \neq 0$
i.e. $\nu_i^* = 0$

in this case ① simplifies to:

$$\frac{g_i/n_i}{1 + P_i^* \, g_i/n_i} = \mu^*$$

$$\mu^{\alpha} = \frac{g_i/n_i}{1 + P_i^{\alpha} \cdot \frac{g_i}{n_i}}$$

$$1 + P_i^{\alpha} \frac{g_i}{n_i} = \frac{g_i/n_i}{\mu^{\alpha}}$$

$$P_i^{\alpha} = \left(\frac{g_i/n_i}{\mu^{\alpha}} - 1 \right) \cdot \frac{1}{\left(\frac{g_i}{n_i}\right)}$$

$$= \frac{1}{\mu^{\alpha}} - \frac{1}{\left(\frac{g_i}{n_i}\right)} =$$

$$\boxed{P_i^{\alpha} = \left(\frac{1}{\mu^{\alpha}} - \frac{n_i}{g_i} \right)^+} \longrightarrow (a)$$

water level.

or

$$P_i^{\alpha} = 0$$

moreover $\sum P_i^{\alpha} = P_T$.

$$\boxed{\sum_{i=1}^{N} \left(\frac{1}{\mu^{\alpha}} - \frac{n_i}{g_i} \right)^+ = P_T} \quad (b)$$

together a, b give the waterfilling solution.

in HW3:

$$R_i = \log\left(1 + P_i \frac{g_i}{n_i}\right)$$

$$\min \sum_{i=1}^{N} \left(R_i - T_i \right)^2$$

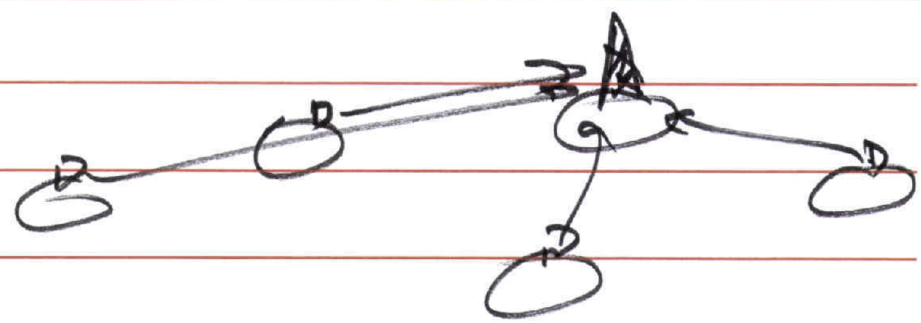
$$\text{s.t.} \quad \sum P_i \leq P_T$$

smallest when $R_i = T_i$

$$\min f(\ ) = \max - f(\ )$$

Slotted Aloha - Randomized Access.

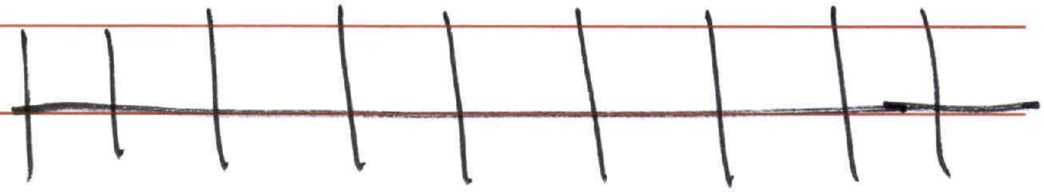
U. Hawaii
early 1970's



Single channel shared by all transmitters

when you have a pkt, send it.

prob. p, iid over time



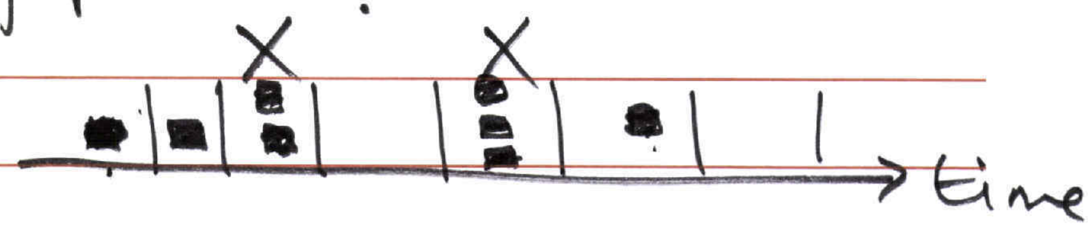
equivalently: they always have a (backlogged) pkt to send, but send with a randomized (prob. p.) (iid) procedure

Backlogged
Saturated
senders

n nodes, backlogged / Saturated

each node indep. transmits w/ prob. p. at each slot (iid over time)

Saturation Throughput = ?

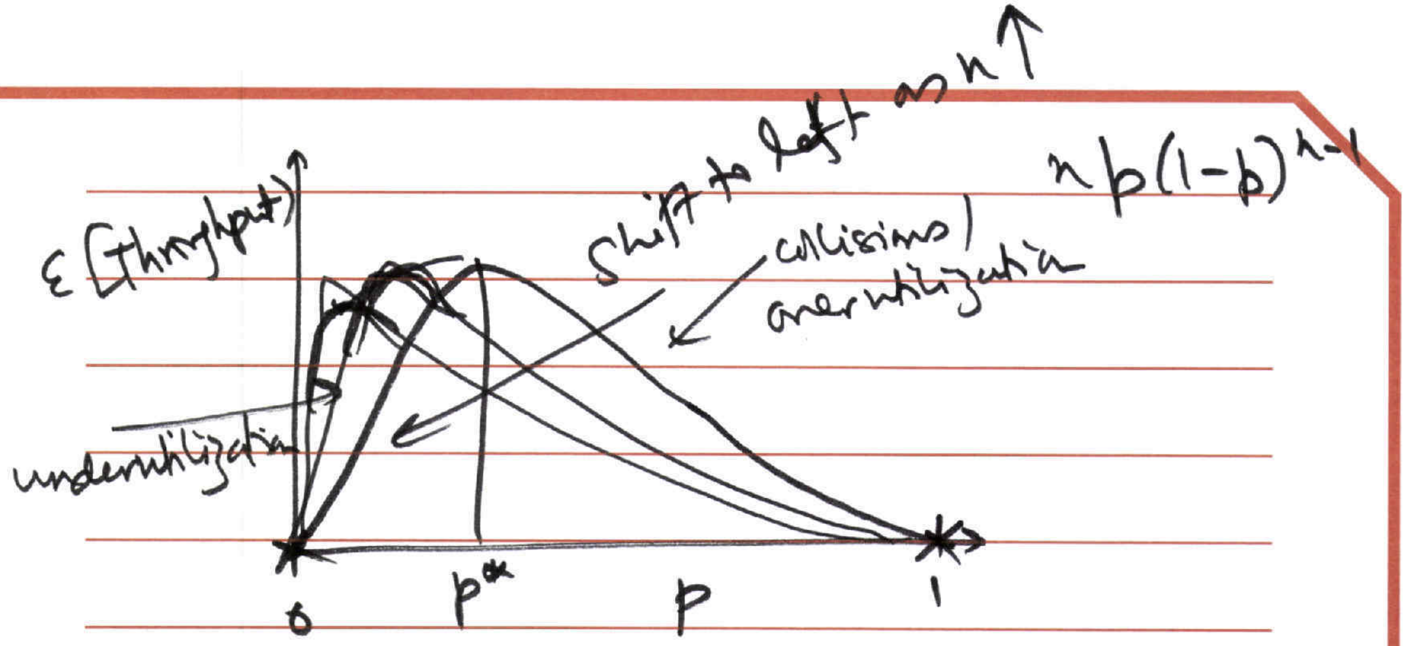


collision if more than 1 pkt in each slot

$E[\text{Throughput}] = ? = np(1-p)^{n-1}$

from in 1 time slot, since iid:

$$\#\text{Packets}: \begin{cases} 0 \text{ pkts} & \boxed{\text{Prob}} \\ & (1-p)^n \\ 1 \text{ pkt} & np(1-p)^{n-1} \\ >1 \text{ pkts} & 1-(1-p)^n - np(1-p)^{n-1} \end{cases}$$



$n = 2.$ $\max_p 2\,p(1-p)$

max when $p^* = 1/2$

$$2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

← max throughput for 2 users.

In general:

$$\max_p n\,p(1-p)^{n-1} = T.$$

for fixed n: $\frac{\partial T}{\partial p} = 0 \Rightarrow \left.\frac{d\left(p(1-p)^{n-1}\right)}{dp}\right|_{p^*} = 0$

$$(1-p^*)^{n-1} + p^* \cdot (n-1) \cdot (1-p^*)^{n-2} \cdot -1 = 0$$

$$\Rightarrow (1-p^*)^{n-1} = p^* \cdot (n-1) \cdot (1-p^*)^{n-2}$$

$$1 - p^* = p^*(n-1)$$

$$= 1 - \cancel{p} = p^* \cdot n - \cancel{p}$$

$$\boxed{p^* = \frac{1}{n}}$$

max throughput: $n \cdot p^* (1 - p^*)^{n-1}$

$$\cancel{n} \cdot \frac{\cancel{1}}{\cancel{n}} \left(1 - \frac{1}{n}\right)^{n-1}$$

$n = 2$ $\quad$ $1/2$

$n = 3$ $\quad$ $4/9$

$\vdots$

$$T^*(n) \longrightarrow \frac{1}{e}$$

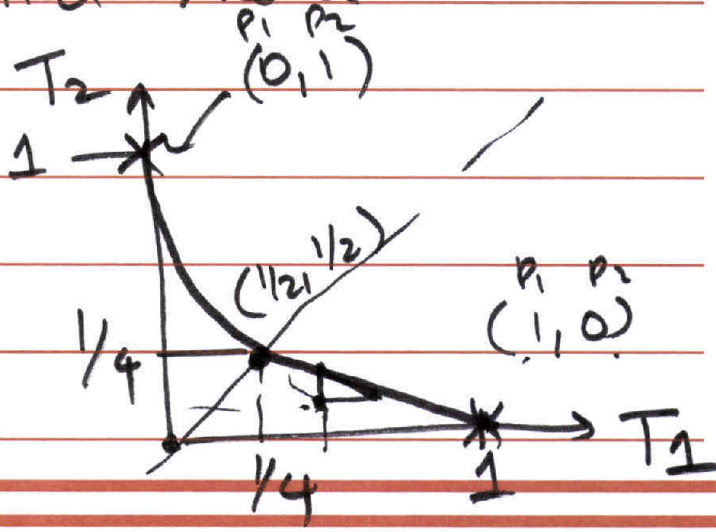
$\sim 36\%$

max throughput of slotted Aloha as $n \to \infty$. (large)

What if different users ~~send~~ need to send different rates?

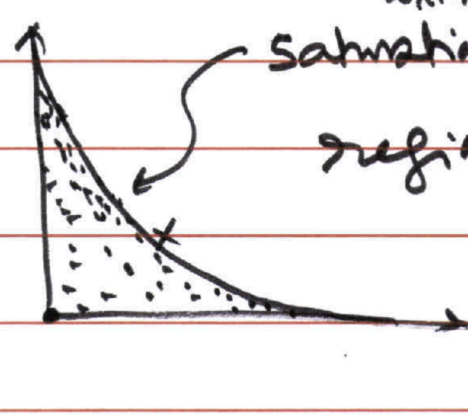
p_i - slotted Aloha:

2 users: p_1, p_2



$$T_1 = p_1 (1 - p_2)$$
$$T_2 = p_2 (1 - p_1)$$

maximum feasible saturation throughput region



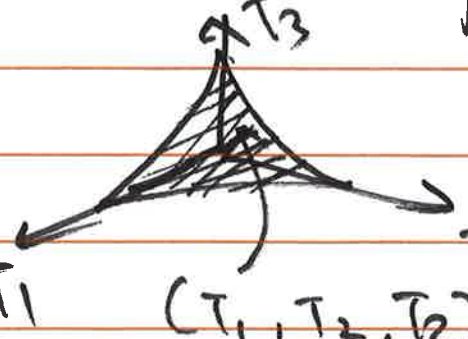
- <u>Key Result</u>:

At the boundary of the 2-user p_i-slotted Aloha saturation throughput region: $p_1 + p_2 = 1$.

— not an OBVIOUS fact!

Is it true that $\sum_{i=1}^{n} p_i = 1$ for any number of users to have a throughput vector at the boundary of the saturation throughput region?

$\boxed{\text{YES!}}$



(T_1, T_2, T_3) on the boundary corresponds to (p_1, p_2, p_3) s.t. $p_1 + p_2 + p_3 = 1$

more general result for p_i-slotted Aloha:

At the boundary of n-user saturation throughput region:

$$\boxed{\sum p_i = 1}$$

$$T_i = p_i \prod_{j \neq i} (1 - p_j)$$

$$\vec{p} \longrightarrow \vec{T}$$

In theory, can now formulate:

$$\boxed{\begin{array}{l} \max f(T_1, T_2, T_3) \\ \text{s.t.} \quad \sum p_i = 1 \end{array}}$$